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Cohomology structure for a Poisson algebra: I

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We introduce quasi-Poisson cohomology groups for a Poisson algebra, which can be computed by its quasi-Poisson complex. Moreover, there exists a Grothendieck spectral sequence relating quasi-Poisson cohomology to Hochschild cohomology and Lie algebra cohomology.

Keywords: Poisson algebra; quasi-Poisson enveloping algebra; quasi-Poisson module; quasi-Poisson cohomology.

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1. Introduction

The aim of this paper is to construct a cohomology theory for Poisson algebras (not necessarily commutative). Here, we follow the notion of Poisson algebras as introduced in [5]. By definition, a *Poisson algebra* over a field $\Bbbk$ consists of a triple $(A, \cdot, \{-,-\})$, where $(A, \cdot)$ is an associative $\Bbbk$-algebra and $(A, \{-,-\})$ is a Lie algebra over $\Bbbk$, such that the Leibniz rule $\{a, bc\} = \{a, b\}c + b\{a, c\}$ holds for all $a, b, c \in A$. As a natural generalization of commutative Poisson algebras, this version of Poisson algebras has been studied by many authors [4, 10–13, 18, 19]. We also stress that there are other different versions of noncommutative Poisson structures, see e.g. [3, 16, 17].

We would like to mention that this version of Poisson algebras is quite useful in the study of commutative Poisson algebras, especially in the deformation theory. In fact, an easy observation shows that a commutative Poisson algebra A has a deformation quantization if and only if A can be deformed to a standard Poisson algebra of some peculiar form within this version of Poisson algebras. Consequently, if A has no nontrivial deformation, then A has no deformation quantization. In some cases, for instance if A is prime as an associative algebra, then any deformation of A will be standard [4, Theorem 1.2], and hence A admits nontrivial deformation if and only if A has (higher version of) deformation quantization. We refer to [1] for more details.

Let A be a Poisson algebra. In [18], the authors introduced the *quasi-Poisson enveloping algebra* $\mathcal{Q}(A)$ for A, which is an associative algebra, and proved that the category of quasi-Poisson modules over A is equivalent to the category of modules over $\mathcal{Q}(A)$, see also Sec. 2 below for detail. Under the Lie bracket, the regular A-bimodule A becomes a quasi-Poisson module over A and hence is a $\mathcal{Q}(A)$-module. We define the *quasi-Poisson cohomology group of A with coefficients in the quasi-Poisson module M* to be the Yoneda-Ext groups $\operatorname{Ext}^*_{\mathcal{Q}(A)}(A, M)$, see Definition 3.1. By constructing a projective resolution of the quasi-Poisson module A, we introduce the *quasi-Poisson complex* to simplify the calculation of the quasi-Poisson cohomology, see Theorem 3.7, Definition 3.9 and Proposition 3.10. Applying this construction, we give explanation of lower dimensional quasi-Poisson cohomology groups. Some examples are calculated in Sec. 4.

In Sec. 5, we show that the quasi-Poisson cohomology is closely related to the Hochschild cohomology and Lie algebra cohomology. In fact, there exists a Grothendieck spectral sequence, connecting the quasi-Poisson cohomology with the Hochschild cohomology and the Lie algebra cohomology, see Theorem 5.4. In some extreme cases, the quasi-Poisson cohomology algebra is shown to be the tensor product of the Hochschild cohomology and the Lie algebra cohomology.

The quasi-Poisson cohomology has an important application in the calculation of so-called Poisson cohomology, which controls the formal deformation of a Poisson algebra, see [1] or [5] for details. In [1], the authors construct a long exact sequence including Poisson cohomology, quasi-Poisson cohomology and Lie algebra cohomology. Another possible application is to the study of an associative algebra, for each associative algebra admits a standard Poisson structure. The quasi-Poisson cohomology group of standard Poisson algebras may give us an interesting invariant. In fact, by Theorem 5.4, the quasi-Poisson cohomology groups carry the information of both the Hochschild cohomology and the Lie algebra cohomology.

Throughout $\mathbb{k}$ will be a field of characteristic zero. All algebras considered are over $\mathbb{k}$ and have a multiplicative identity element. We write $\otimes = \otimes_{\mathbb{k}}$ and $\operatorname{Hom} = \operatorname{Hom}_{\mathbb{k}}$ for brevity.

2. Preliminaries

Let $(A, \cdot, \{-,-\})$ be a Poisson algebra (not necessarily commutative). A *quasi-Poisson A-module* is an A-bimodule M together with a $\Bbbk$-bilinear map $\{-,-\}_* : A \times M \to M$, which satisfies

$$\begin{aligned}\{a, bm\}_* &= \{a, b\}m + b\{a, m\}_*,\\ \{a, mb\}_* &= m\{a, b\} + \{a, m\}_* b,\\ \{\{a, b\}, m\}_* &= \{a, \{b, m\}_*\}_* - \{b, \{a, m\}_*\}_*\end{aligned}$$

for any $a, b \in A$ and $m \in M$. Let M, N be quasi-Poisson modules. A *homomorphism* of quasi-Poisson A-modules is a $\Bbbk$-linear map $f\colon M \to N$ which is a homomorphism of both A-bimodules and Lie modules.

Let us recall the definition of *quasi-Poisson enveloping algebras*, see [18] for more details. We need some convention.

Denote by A^{op} the opposite algebra of the associative algebra A. To avoid confusion, we usually use a to denote an element in A and a' its counterpart in A^{op}. The algebra $A^e = A \otimes A^{\mathrm{op}}$ is called the *enveloping algebra* of the associative algebra A. Denote by $\mathcal{U}(A)$ the *universal enveloping algebra* of the Lie algebra $(A, \{-,-\})$. It is well known that the category of A-bimodules is isomorphic to the category of left modules over A^e and the category of Lie modules over A is isomorphic to the category of left $\mathcal{U}(A)$-modules. Note that the universal enveloping algebra $\mathcal{U}(A)$ is a cocommutative Hopf algebra. Using Sweedler's notation, the comultiplication is denoted by $\Delta(X) = \sum X_1 \otimes X_2$ for any $X \in \mathcal{U}(A)$. The counit map is denoted by ϵ and the identity element of $\mathcal{U}(A)$ is denoted by $\mathbb{1}$. The Lie bracket makes A a Lie module over A, or equivalently, a $\mathcal{U}(A)$-module with the action given by

$$X(a) = \{x_1, \{x_2, \ldots, \{x_n, a\}\} \cdots\}$$

and $\mathbb{1}(a) = a$ for any $X = x_1 \otimes x_2 \otimes \cdots \otimes x_n \in \mathcal{U}(A), n \geq 1$ and $a \in A$. Hence the usual tensor product makes A^e a $\mathcal{U}(A)$-module with the action given by

$$X(a \otimes b') = \sum X_1(a) \otimes (X_2(b))'$$

for $X \in \mathcal{U}(A), a \otimes b' \in A^e$. Moreover, by the co-commutativity of the co-product of $\mathcal{U}(A)$, we know that A^e is a $\mathcal{U}(A)$-module algebra, which means that the multiplication $A^e \otimes A^e \to A^e$ is a $\mathcal{U}(A)$-homomorphism. Then, we obtain a smash product $A^e \# \mathcal{U}(A)$, which is an associative algebra, see [15, Sec. 7.2]. Recall that $A^e \# \mathcal{U}(A) = A^e \otimes \mathcal{U}(A)$ as a $\Bbbk$-vector space and the multiplication is given by

$$(a \otimes b' \# X)(c \otimes d' \# Y) = \sum a X_1(c) \otimes (X_2(d)b)' \# X_3 Y,$$

and the identity element is $1_A \otimes 1'_A \# \mathbb{1}$, where 1_A is the multiplicative identity of A. For more details, we refer to [18].

Definition 2.1 ([18]). Let $A = (A, \cdot, \{-,-\})$ be a Poisson algebra. The smash product $A^e \# \mathcal{U}(A)$ is called the *quasi-Poisson enveloping algebra* of A and denoted by $\mathcal{Q}(A)$.

Remark 2.2. By definition, the quasi-Poisson enveloping algebra $\mathcal{Q}(A)$ is just $A \otimes A^{\mathrm{op}} \otimes \mathcal{U}(A)$ as a $\Bbbk$-vector space. Let $\{v_i \,|\, i \in S\}$ be a $\Bbbk$-basis for A, where S is an index set with a total ordering $\leq$. Thus $\mathcal{Q}(A)$ has a PBW-basis given by

$$\{v_i \otimes v_j' \# (v_{k_1} \otimes \cdots \otimes v_{k_r}) \,|\, i, j, k_1, \ldots, k_r \in S, k_1 \leq \cdots \leq k_r, r \geq 0\}.$$

Theorem 2.3 ([18]). *The category of quasi-Poisson modules over A is isomorphic to the category of left $\mathcal{Q}(A)$-modules.*

Given a quasi-Poisson A-module M, one can define a $\mathcal{Q}(A)$-module M by setting

$$(a \otimes b' \# X)m = aX(m)b$$

for all $m \in M$ and $a \otimes b' \# X \in \mathcal{Q}(A)$. Conversely, given a left $\mathcal{Q}(A)$-module M, we set

$$am = (a \otimes 1_A' \# \mathbb{1})m, \quad ma = (1_A \otimes a' \# \mathbb{1})m, \quad \{a, m\}_* = (1_A \otimes 1_A' \# a)m$$

for all $m \in M, a \in A$ to obtain a quasi-Poisson module over A.

3. Quasi-Poisson Cohomology

By Theorem 2.3, we know that there are enough projective and injective objects in the category of quasi-Poisson modules. Consequently, one can construct a cohomology theory for a Poisson algebra by using projective or injective resolutions in a standard way. In fact, under the action $\{-,-\}_* = \{-,-\}$, the regular A-bimodule A becomes a quasi-Poisson module over A. Then we may consider the Yoneda-Ext groups $\mathrm{Ext}^*_{\mathcal{Q}(A)}(A, M)$ for any quasi-Poisson module M.

Definition 3.1. Let A be a Poisson algebra and $\mathcal{Q}(A)$ the quasi-Poisson enveloping algebra of A. For any quasi-Poisson module M, the extension group $\mathrm{Ext}^n_{\mathcal{Q}(A)}(A, M)$ is called the nth *quasi-Poisson cohomology group of A with coefficients in M*, and denoted by $\mathrm{HQ}^n(A, M)$.

Remark 3.2. The extension group $\mathrm{HQ}^n(A, A)$ is simply denoted by $\mathrm{HQ}^n(A)$. One may consider the Yoneda-Ext algebra $\mathrm{HQ}^*(A) = \bigoplus_{n \geq 0} \mathrm{HQ}^n(A)$ with the multiplication given by the Yoneda product, which is called the *quasi-Poisson cohomology algebra* of A. Clearly, $\mathrm{HQ}^*(A)$ is non-negatively graded and each $\mathrm{HQ}^*(A, M)$ is a graded right $\mathrm{HQ}^*(A)$-module.

3.1. *A free resolution of A as a $\mathcal{Q}(A)$-module*

In the sequel, we will construct a projective resolution of A as a $\mathcal{Q}(A)$-module, so that we can compute the cohomology groups $\mathrm{Ext}^n_{\mathcal{Q}(A)}(A, M)$ in a standard way.

For convenience, for each $i, j \geq 0$, we denote by A^i and $\wedge^j$ the ith tensor product and the jth exterior power of the $\Bbbk$-space A, respectively.

Our construction is based on the following two well-known resolutions. One is the bar resolution of A as an A^e-module (A-bimodule)

$$\mathbb{S}\colon\ \cdots \to A^{i+2} \xrightarrow{\delta_i} A^{i+1} \to \cdots \to A \otimes A \otimes A \xrightarrow{\delta_1} A \otimes A \to 0,$$

with the differential

$$\delta_i(a_0 \otimes a_1 \otimes \cdots \otimes a_{i+1}) = \sum_{k=0}^{i} (-1)^k a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_{i+1},$$

for any $a_0 \otimes a_1 \otimes \cdots \otimes a_{i+1} \in A^{i+2}, i \geq 1$. The other one is the projective resolution of $\Bbbk$ as a trivial $\mathcal{U}(A)$-module

$$\mathbb{K} : \cdots \to \mathcal{U}(A) \otimes \wedge^j \xrightarrow{d_j} \mathcal{U}(A) \otimes \wedge^{j-1} \to \cdots \to \mathcal{U}(A) \otimes \wedge^1 \xrightarrow{d_1} \mathcal{U}(A) \to 0,$$

with the differential

$$\begin{aligned} &d_j(X \otimes x_1 \wedge \cdots \wedge x_j) \\ &\quad = \sum_{l=1}^{j} (-1)^{l+1} (X \cdot x_l) \otimes (x_1 \wedge \cdots \widehat{x}_l \cdots \wedge x_j) \\ &\qquad + \sum_{1 \leq p < q \leq j} (-1)^{p+q} X \otimes (\{x_p, x_q\} \wedge x_1 \wedge \cdots \widehat{x}_p \cdots \widehat{x}_q \cdots \wedge x_j), \end{aligned}$$

where $X \cdot x_l$ is the product of X and x_l in $\mathcal{U}(A)$, and the symbol $\widehat{x}_l$ indicates that the term x_l is to be omitted.

Taking the tensor product of $\mathbb{S}$ and $\mathbb{K}$, we obtain the following bicomplex

$$\begin{tikzcd} & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\ 0 & A^4 \otimes \mathcal{U}(A) \arrow[l] \arrow[d] & A^4 \otimes \mathcal{U}(A) \otimes \wedge^1 \arrow[l] \arrow[d] & A^4 \otimes \mathcal{U}(A) \otimes \wedge^2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ 0 & A^3 \otimes \mathcal{U}(A) \arrow[l] \arrow[d] & A^3 \otimes \mathcal{U}(A) \otimes \wedge^1 \arrow[l] \arrow[d] & A^3 \otimes \mathcal{U}(A) \otimes \wedge^2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ 0 & A^2 \otimes \mathcal{U}(A) \arrow[l] \arrow[d] & A^2 \otimes \mathcal{U}(A) \otimes \wedge^1 \arrow[l] \arrow[d] & A^2 \otimes \mathcal{U}(A) \otimes \wedge^2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ & 0 & 0 & 0 & \end{tikzcd}$$

and its total complex $\mathrm{Tot}(\mathbb{S} \otimes \mathbb{K})$ is written as

$$\mathbb{Q} : \cdots \to Q_n \xrightarrow{\varphi_n} Q_{n-1} \to \cdots \to Q_1 \xrightarrow{\varphi_1} Q_0 \to 0, \tag{3.1}$$

where $Q_0 = A^2 \otimes \mathcal{U}(A)$, and for $n \geq 1$,

$$Q_n = \bigoplus_{i+j=n} A^{i+2} \otimes \mathcal{U}(A) \otimes \wedge^j,$$

$$\varphi_n = \bigoplus_{i+j=n} (\delta_i \otimes \mathrm{id} + (-1)^i \mathrm{id} \otimes d_j).$$

The following lemmas will be handy for later use. Some of them seem to be well known. For the convenience of the reader, we also include a proof.

Lemma 3.3. *Let H be a Hopf algebra and B an H-module algebra. Let M be a $B\#H$-module and N an H-module. Then $M\otimes N$ becomes a $B\#H$-module via the action*

$$(b\#h)(m\otimes n)=\sum b(h_1m)\otimes h_2n,$$

for any $b\#h\in B\#H, m\otimes n\in M\otimes N$.

Proof. By definition, it suffices to check the equality

$$(b\#h)((b'\#h')(m\otimes n))=((b\#h)(b'\#h'))(m\otimes n) \tag{3.2}$$

for any $b\#h, b'\#h'\in B\#H$ and $m\otimes n\in M\otimes N$. Indeed,

$$\begin{aligned}&(b\#h)((b'\#h')(m\otimes n))\\&\quad=(b\#h)\left(\sum b'(h'_1m)\otimes h'_2n\right)\\&\quad=\sum bh_1(b'(h'_1m))\otimes h_2(h'_2n)\\&\quad=\sum b(h_{11}b')(h_{12}h'_1m)\otimes h_2(h'_2n),\end{aligned}$$

where the last equality is deduced from the assumption that M is a $B\#H$-module. On the other hand, we have

$$\begin{aligned}&((b\#h)(b'\#h'))(m\otimes n)\\&\quad=\left(\sum b(h_1b')\#h_2h'\right)(m\otimes n)\\&\quad=\sum b(h_1b')((h_2h')_1m)\otimes(h_2h')_2n\\&\quad=\sum b(h_1b')(h_{21}h'_1m)\otimes h_{22}h'_2n,\end{aligned}$$

and the desired equality (3.2) holds. □

Taking $H=\mathcal{U}(A), B=A^e, M=A^i$ $(i\geq 1)$ and $N=\mathcal{U}(A)$, we observe that A^i is an $A^e\#\mathcal{U}(A)$-module with the action given by

$$(a\otimes b'\#X)(a_1\otimes\cdots\otimes a_i)=\sum aX_1(a_1)\otimes\cdots\otimes X_i(a_i)b$$

for any $a\otimes b'\#X\in A^e\#\mathcal{U}(A), a_1\otimes\cdots\otimes a_i\in A^i$. In fact, we have

$$\begin{aligned}&(1_A\otimes 1'_A\#X)((a\otimes b'\#\mathbb{1})(a_1\otimes\cdots\otimes a_i))\\&\quad=\sum X(aa_1\otimes a_2\otimes\cdots\otimes a_{i-1}\otimes a_ib)\\&\quad=\sum X_1(aa_1)\otimes X_2(a_2)\otimes\cdots\otimes X_{i-1}(a_{i-1})\otimes X_i(a_ib)\end{aligned}$$

$$
\begin{aligned}
&= \sum (X_{11}(a)X_{12}(a_1)) \otimes X_2(a_2) \otimes \cdots \otimes X_{i-1}(a_{i-1}) \otimes (X_{i1}(a_i)X_{i2}(b)) \\
&= \sum ((X_{11}(a) \otimes (X_{i2}(b))'\#\mathbb{1})(X_{12}(a_1) \otimes X_2(a_2) \\
&\quad \otimes \cdots \otimes X_{i-1}(a_{i-1}) \otimes X_{i1}(a_i))) \\
&= ((1_A \otimes 1'_A\#X)(a \otimes b'\#\mathbb{1}))(a_1 \otimes \cdots \otimes a_i)
\end{aligned}
$$

where the third equality is deduced from the Leibniz rule and the last one from the co-commutativity of $\mathcal{U}(A)$. By Lemma 3.3, we get a $\mathcal{Q}(A)$-module $A^i \otimes \mathcal{U}(A)$ with the action given by

$$
\begin{aligned}
&(a \otimes b'\#X)(a_1 \otimes a_2 \otimes \cdots \otimes a_i \otimes Y) \\
&\quad := \sum aX_1(a_1) \otimes X_2(a_2) \otimes \cdots \otimes X_{i-1}(a_{i-1}) \otimes X_i(a_i)b \otimes X_{i+1}Y
\end{aligned}
$$

for all $a \otimes b'\#X \in \mathcal{Q}(A)$, $a_1 \otimes \cdots \otimes a_i \otimes Y \in A^i \otimes \mathcal{U}(A)$. Moreover, for any $j \geq 0$, $A^i \otimes \mathcal{U}(A) \otimes \wedge^j$ is a $\mathcal{Q}(A)$-module with the $\mathcal{Q}(A)$-action induced from the one on $A^i \otimes \mathcal{U}(A)$.

Lemma 3.4. *For any $i, j \geq 0$, $A^{i+2} \otimes \mathcal{U}(A) \otimes \wedge^j$ is a free $\mathcal{Q}(A)$-module.*

Proof. It suffices to prove that $A^{i+2} \otimes \mathcal{U}(A)$ is a free module, since $A^{i+2} \otimes \mathcal{U}(A) \otimes \wedge^j$ is a direct sum of copies of $A^{i+2} \otimes \mathcal{U}(A)$.

Choose a $\mathbb{k}$-basis $\{v_i \mid i \in S\}$ for A, where S is an index set with total ordering $\leq$, and we claim that $A^{i+2} \otimes \mathcal{U}(A)$ is a free $\mathcal{Q}(A)$-module with a basis

$$\{1_A \otimes v_{k(1)} \otimes \cdots \otimes v_{k(i)} \otimes 1_A \otimes \mathbb{1} \mid k(1), \ldots, k(i) \in S, i \geq 0\}.$$

Note that there exists a PBW-basis of $\mathcal{Q}(A)$ given by $v_s \otimes v'_t\#(v_{i(1)} \otimes \cdots \otimes v_{i(r)})$ with $s, t, i(1), \ldots, i(r) \in S$ and $i(1) \leq \cdots \leq i(r), r \geq 0$. Following the notation in [18], we write $\overrightarrow{\alpha} = v_{i(1)} \otimes \cdots \otimes v_{i(r)}$ in $\mathcal{U}(A)$ for $\alpha = (i(1), \ldots, i(r)) \in S^r$, $i(1) \leq \cdots \leq i(r)$ and call r the degree of the element $\overrightarrow{\alpha}$. Similarly, we also denote $\overrightarrow{\theta} = v_{k(1)} \otimes \cdots \otimes v_{k(i)}$ in A^i for $\theta = (k(1), \ldots, k(i)) \in S^i$.

Assume that some $\mathcal{Q}(A)$-linear combination equals to zero, that is,

$$\sum_{s,t,\alpha,\theta} \lambda_{s,t,\alpha,\theta}(v_s \otimes v'_t\#\overrightarrow{\alpha})(1_A \otimes \overrightarrow{\theta} \otimes 1_A \otimes \mathbb{1}) = 0,$$

where each $v_s \otimes v'_t\#\overrightarrow{\alpha}$ is chosen to be in the PBW-basis.

Each term in the left-hand side is written as

$$
\begin{aligned}
&(v_s \otimes v'_t\#\overrightarrow{\alpha})(1_A \otimes \overrightarrow{\theta} \otimes 1_A \otimes \mathbb{1}) \\
&\qquad = \sum v_s \otimes \overrightarrow{\alpha_1}(\overrightarrow{\theta}) \otimes v_t \otimes \overrightarrow{\alpha_2} \\
&\qquad = \sum v_s \otimes \overrightarrow{\theta} \otimes v_t \otimes \overrightarrow{\alpha} + \cdots,
\end{aligned}
$$

where each $\overrightarrow{\alpha_2}$ in the rest terms has degree strictly less than the degree of α. Let n be the highest degree of α's occurring in the above sum. Combining those terms with $\deg(\overrightarrow{\alpha}) = n$ in the resulting sum, we have

$$\sum_{\substack{s,t,\alpha,\theta \\ \deg(\alpha)=n}} \lambda_{s,t,\alpha,\theta}(v_s \otimes \overrightarrow{\theta} \otimes v_t \otimes \overrightarrow{\alpha}) = 0.$$

Thus $\lambda_{s,t,\alpha,\theta} = 0$ for any s, t, α, θ, which completes the proof. $\square$

Lemma 3.5. *The morphism φ_n $(n \geq 0)$ in the total complex* (3.1) *is a $\mathcal{Q}(A)$-homomorphism.*

Proof. By definition, each φ_n in $\mathbb{Q}$ is a direct sum of $\binom{\delta_i \otimes \mathrm{id}}{(-1)^i \mathrm{id} \otimes d_j}$. It suffices to show that both $\delta_i \otimes \mathrm{id}$ and $\mathrm{id} \otimes d_j$ are homomorphisms of $\mathcal{Q}(A)$-modules. In fact, $\delta_i \otimes \mathrm{id}$ and $\mathrm{id} \otimes d_j$ are A^e-homomorphisms and hence

$$(\delta_i \otimes \mathrm{id})((a \otimes b'\#\mathbb{1})x) = (a \otimes b'\#\mathbb{1})(\delta_i \otimes \mathrm{id})(x),$$
$$(\mathrm{id} \otimes d_j)((a \otimes b'\#\mathbb{1})x) = (a \otimes b'\#\mathbb{1})(\mathrm{id} \otimes d_j)(x),$$

for all $x \in A^{i+2} \otimes \mathcal{U}(A) \otimes \wedge^j$.

On the other hand, for any $1_A \otimes 1'_A\#X \in \mathcal{Q}(A)$, $a_1 \otimes \cdots \otimes a_{i+2} \otimes Y \otimes \omega^j \in A^{i+2} \otimes \mathcal{U}(A) \otimes \wedge^j$, we have

$$\begin{aligned}
&(\delta_i \otimes \mathrm{id})((1_A \otimes 1'_A\#X)(a_1 \otimes \cdots \otimes a_{i+2} \otimes Y \otimes \omega^j)) \\
&\quad = (\delta_i \otimes \mathrm{id})\left(\sum X_1(a_1) \otimes \cdots \otimes X_{i+2}(a_{i+2}) \otimes X_{i+3}Y \otimes \omega^j\right) \\
&\quad = \sum_{k=1}^{i+1} \sum (-1)^{k-1} X_1(a_1) \otimes \cdots \otimes X_k(a_k)X_{k+1}(a_{k+1}) \\
&\qquad \otimes \cdots \otimes X_{i+2}(a_{i+2}) \otimes X_{i+3}Y \otimes \omega^j \\
&\quad = \sum_{k=1}^{i+1} \sum (-1)^{k-1} X_1(a_1) \otimes \cdots \otimes X_k(a_k a_{k+1}) \\
&\qquad \otimes \cdots \otimes X_{i+1}(a_{i+2}) \otimes X_{i+2}Y \otimes \omega^j \\
&\quad = (1_A \otimes 1'_A\#X)\left(\sum_{k=1}^{i+1} (-1)^{k-1} a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_{i+2} \otimes Y \otimes \omega^j\right) \\
&\quad = (1_A \otimes 1'_A\#X)((\delta_i \otimes \mathrm{id})(a_1 \otimes \cdots \otimes a_{i+2} \otimes Y \otimes \omega^j)),
\end{aligned}$$

where the third equality is deduced from the Leibniz rule. By the definition of d_j, it is easy to check that

$$\begin{aligned}
&(\mathrm{id} \otimes d_j)((1_A \otimes 1'_A\#X)(a_1 \otimes \cdots \otimes a_{i+2} \otimes Y \otimes \omega^j)) \\
&\quad = (1_A \otimes 1'_A\#X)((\mathrm{id} \otimes d_j)(a_1 \otimes \cdots \otimes a_{i+2} \otimes Y \otimes \omega^j)).
\end{aligned}$$

Since the quasi-Poisson enveloping algebra $\mathcal{Q}(A)$ can be generated by the elements $a \otimes b'\#\mathbb{1}$ and $1_A \otimes 1'_A\#X$ for all $a, b \in A, X \in \mathcal{U}(A)$, we have that $\delta_i \otimes \mathrm{id}$ and $\mathrm{id} \otimes d_j$ are $\mathcal{Q}(A)$-homomorphisms. $\square$

Lemma 3.6. *Keeping the above notation, we have*

$$\mathrm{H}_0(\mathbb{Q}) \cong A \quad \textit{and} \quad \mathrm{H}_n(\mathbb{Q}) = 0, \ \ n \geq 1.$$

Proof. By Künneth's theorem (see [8, Chap. V, Theorem 2.1]), it is easily seen that $\mathbb{Q}$ is exact at Q_n for each $n \geq 1$, since both $\mathbb{S}$ and $\mathbb{K}$ are exact for $i, j > 0$ and $\mathbb{Q}$ is the total complex of $\mathbb{S} \otimes \mathbb{K}$. For $n = 0$, applying the Künneth's theorem again, we get

$$H_0(\mathbb{Q}) \cong H_0(\mathbb{S}) \otimes H_0(\mathbb{K}) = A \otimes \Bbbk \cong A. \qquad \square$$

Combining Lemmas 3.4, 3.5 and 3.6, we obtain a projective resolution of A as a $\mathcal{Q}(A)$-module.

Theorem 3.7. *Let A be a Poisson algebra and $\mathcal{Q}(A)$ the quasi-Poisson enveloping algebra of A. Then the complex $\mathbb{Q}$ together with the $\mathcal{Q}(A)$-homomorphism $\varphi_0 \colon Q_0 \to A$ given by $\varphi_0(a_0 \otimes a_1 \otimes X) = \epsilon(X)a_0a_1$ is a free resolution of A as a $\mathcal{Q}(A)$-module, where ϵ is the counit map of $\mathcal{U}(A)$.*

Let M be a left $\mathcal{Q}(A)$-module. Applying the functor $\mathrm{Hom}_{\mathcal{Q}(A)}(-, M)$ to the complex $\mathbb{Q}$, we obtain a complex $\mathrm{Hom}_{\mathcal{Q}(A)}(\mathbb{Q}, M)$:

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{Q}(A)}(Q_0, M) &\to \mathrm{Hom}_{\mathcal{Q}(A)}(Q_1, M) \to \mathrm{Hom}_{\mathcal{Q}(A)}(Q_2, M) \to \cdots \\ &\to \mathrm{Hom}_{\mathcal{Q}(A)}(Q_n, M) \to \mathrm{Hom}_{\mathcal{Q}(A)}(Q_{n+1}, M) \to \cdots. \end{aligned}$$

By Theorem 3.7, the nth quasi-Poisson cohomology group is calculated by

$$\mathrm{HQ}^n(A, M) = \mathrm{Ext}^n_{\mathcal{Q}(A)}(A, M) = H^n\mathrm{Hom}_{\mathcal{Q}(A)}(\mathbb{Q}, M).$$

3.2. *Quasi-Poisson complex*

To compute the quasi-Poisson cohomology groups, one can use a simplified complex, called the *quasi-Poisson complex*. Let M be a quasi-Poisson module. Applying the functor $\mathrm{Hom}_{\mathcal{Q}(A)}(-, M)$ to the bicomplex $\mathbb{S} \otimes \mathbb{K}$, we obtain

$$\begin{tikzcd}
 & \cdots & \cdots & \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{Q}(A)}(A^4 \otimes \mathcal{U}(A), M) \arrow[r] \arrow[u] & \mathrm{Hom}_{\mathcal{Q}(A)}(A^4 \otimes \mathcal{U}(A) \otimes \wedge^1, M) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{Q}(A)}(A^3 \otimes \mathcal{U}(A), M) \arrow[r] \arrow[u] & \mathrm{Hom}_{\mathcal{Q}(A)}(A^3 \otimes \mathcal{U}(A) \otimes \wedge^1, M) \arrow[r] \arrow[u] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_{\mathcal{Q}(A)}(A^2 \otimes \mathcal{U}(A), M) \arrow[r] \arrow[u] & \mathrm{Hom}_{\mathcal{Q}(A)}(A^2 \otimes \mathcal{U}(A) \otimes \wedge^1, M) \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Following from the natural $\Bbbk$-isomorphisms

$$\begin{aligned}
&\Phi_{i,j}\colon\ \mathrm{Hom}_{\mathcal{Q}(A)}(A^{i+2}\otimes\mathcal{U}(A)\otimes\wedge^j, M)\xrightarrow{\simeq}\mathrm{Hom}(A^i\otimes\wedge^j, M),\\
&\Phi_{i,j}(f)((a_1\otimes\cdots\otimes a_i)\otimes(x_1\wedge\cdots\wedge x_j))\\
&\qquad= f(1_A\otimes(a_1\otimes\cdots\otimes a_i)\otimes 1_A\otimes\mathbb{1}\otimes(x_1\wedge\cdots\wedge x_j)),
\end{aligned}$$

the above bicomplex is isomorphic to the bicomplex $\mathrm{Hom}(A^\bullet\otimes\wedge^\bullet, M)$:

$$\begin{tikzcd}
 & \cdots & \cdots & \cdots & \\
0 \arrow[r] & \mathrm{Hom}(A^2,M) \arrow[u] \arrow[r, "\sigma_H^{1,0}"] & \mathrm{Hom}(A^2\otimes\wedge^1,M) \arrow[u] \arrow[r, "\sigma_H^{1,1}"] & \mathrm{Hom}(A^2\otimes\wedge^2,M) \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & \mathrm{Hom}(A,M) \arrow[u, "\sigma_V^{1,0}"] \arrow[r, "\sigma_H^{1,0}"] & \mathrm{Hom}(A\otimes\wedge^1,M) \arrow[u, "\sigma_V^{1,1}"] \arrow[r, "\sigma_H^{1,1}"] & \mathrm{Hom}(A\otimes\wedge^2,M) \arrow[u, "\sigma_V^{1,2}"] \arrow[r] & \cdots \\
0 \arrow[r] & M \arrow[u, "\sigma_V^{0,0}"] \arrow[r, "\sigma_H^{0,0}"] & \mathrm{Hom}(\wedge^1,M) \arrow[u, "\sigma_V^{0,1}"] \arrow[r, "\sigma_H^{0,1}"] & \mathrm{Hom}(\wedge^2,M) \arrow[u, "\sigma_V^{0,2}"] \arrow[r] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where

$$\begin{aligned}
&(\sigma_V^{i,j}(f))((a_1\otimes\cdots\otimes a_{i+1})\otimes(x_1\wedge\cdots\wedge x_j))\\
&\quad= a_1 f((a_2\otimes\cdots\otimes a_i)\otimes(x_1\wedge\cdots\wedge x_j))\\
&\qquad+\sum_{k=1}^{i}(-1)^k f((a_1\otimes\cdots\otimes a_k a_{k+1}\otimes\cdots\otimes a_{i+1})\otimes(x_1\wedge\cdots\wedge x_j))\\
&\qquad+(-1)^{i+1} f((a_1\otimes\cdots\otimes a_i)\otimes(x_1\wedge\cdots\wedge x_j))a_{i+1},\\
&(\sigma_H^{i,j}(f))((a_1\otimes\cdots\otimes a_i)\otimes(x_1\wedge\cdots\wedge x_{j+1}))\\
&\quad=\sum_{l=1}^{j+1}(-1)^{l-1}\Bigg[\{x_l, f((a_1\otimes\cdots\otimes a_i)\otimes(x_1\wedge\cdots\widehat{x}_l\cdots\wedge x_{j+1}))\}_*\\
&\qquad-\sum_{t=1}^{i} f((a_1\otimes\cdots\otimes\{x_l,a_t\}\otimes\cdots\otimes a_i)\otimes(x_1\wedge\cdots\widehat{x}_l\cdots\wedge x_{j+1}))\Bigg]\\
&\qquad+\sum_{1\le p<q\le j+1}(-1)^{p+q} f((a_1\otimes\cdots\otimes a_i)\\
&\qquad\otimes(\{x_p,x_q\}\wedge x_1\wedge\cdots\widehat{x}_p\cdots\widehat{x}_q\cdots\wedge x_{j+1}))
\end{aligned}$$

for all $f\in\mathrm{Hom}(A^i\otimes\wedge^j, A)$, and $(a_1\otimes\cdots a_i)\otimes(x_1\wedge\cdots\wedge x_j)\in A^i\otimes\wedge^j$, $i,j\ge 0$.

Remark 3.8. Write $\delta^n=\sigma_V^{n,0}$ and $d^n=\sigma_H^{0,n}$ for each $n\ge 0$. Clearly, the leftmost column $(\mathrm{Hom}(A^\bullet, M),\delta^\bullet)$ is just the Hochschild complex $\mathrm{HC}(A,M)$ (see [7, 9]), and the bottom row $\mathrm{LC}(A, M) = (\mathrm{Hom}(\wedge^\bullet, M), d^\bullet)$ is just Chevalley–Eilenberg complex, which calculates the Lie algebra cohomology $\mathrm{Ext}^*_{\mathcal{U}(A)}(\Bbbk, M)$.

Let $\mathrm{HH}^n(A, M)$ denote the nth Hochschild cohomology of A with coefficients in the A-bimodule M. Let $\mathrm{HL}^n(A, M) = \mathrm{Ext}^n_{\mathcal{U}(A)}(\Bbbk, M)$ denote the nth Lie algebra cohomology of the Lie algebra A with coefficients in the Lie module M. Thus $\mathrm{HH}^n(A, M) = H^n(\mathrm{HC}(A, M))$ and $\mathrm{HL}^n(A, M) = H^n(\mathrm{LC}(A, M))$. In particular, if $M = A$, $\mathrm{HH}^n(A, A)$ and $\mathrm{HL}^n(A, A)$ are simply denoted by $\mathrm{HH}^i(A)$ and $\mathrm{HL}^n(A)$, respectively.

Definition 3.9. Let A be a Poisson algebra and M a quasi-Poisson module. The total complex of $\mathrm{Hom}(A^\bullet \otimes \wedge^\bullet, M)$,

$$0 \to M \xrightarrow{\sigma^0} \mathrm{Hom}(A \oplus \wedge^1, M) \xrightarrow{\sigma^1} \mathrm{Hom}(A^2 \oplus A \otimes \wedge^1 \oplus \wedge^2, M) \xrightarrow{\sigma^2} \cdots$$
$$\to \mathrm{Hom}\left(\bigoplus_{i+j=n} A^i \otimes \wedge^j, M\right) \xrightarrow{\sigma^n} \mathrm{Hom}\left(\bigoplus_{i+j=n+1} A^i \otimes \wedge^j, M\right) \to \cdots$$

with the differential $\sigma^n = \bigoplus_{i+j=n}(\sigma_V^{i,j} + (-1)^i \sigma_H^{i,j})$ is called the *quasi-Poisson complex of A with coefficients in M*, and denoted by $\mathrm{QC}(A, M)$.

An immediate consequence follows.

Proposition 3.10. *The quasi-Poisson complex* $\mathrm{QC}(A, M)$ *is isomorphic to the complex* $\mathrm{Hom}_{\mathcal{Q}(A)}(\mathbb{Q}, M)$, *and hence* $\mathrm{H}^n(\mathrm{QC}(A, M)) = \mathrm{HQ}^n(A, M)$ *for all* $n \geq 0$.

3.3. *Lower-dimensional quasi-Poisson cohomology groups*

First examples are lower-dimensional quasi-Poisson cohomology groups of a Poisson algebra A. We denote by $Z(A)$ and $Z\{A\}$ the centers of the associative algebra and the Lie algebra, respectively. Then we have the following easy result.

Proposition 3.11. *Keep the above notation. Then* $\mathrm{HQ}^0(A) = Z(A) \cap Z\{A\}$.

Denote by $\mathrm{Der}(A)$ and $\mathrm{Der}_L(A)$ the $\Bbbk$-space of associative derivations and the space of Lie derivations, respectively. Consider the maps $\mathrm{ad}\colon A \to \mathrm{Der}(A)$ and $\mathrm{ad}_L\colon A \to \mathrm{Der}_L(A)$ given by $\mathrm{ad}(a) = [-, a]$ and $\mathrm{ad}_L(a) = \{-, a\}$ for all $a \in A$. Consequently, the differential $\sigma^0 = (\mathrm{ad}, \mathrm{ad}_L)$.

Moreover, for any $f = (f_1, f_0) \in \mathrm{Ker}\,\sigma^1$, by Proposition 3.10, we know that $f_1 \in \mathrm{Der}(A)$ and $f_0 \in \mathrm{Der}_L(A)$ and the equality

$$f_1(\{x, a\}) - \{x, f_1(a)\} = f_0(x)a - af_0(x) \tag{3.3}$$

holds for any $(a, x) \in A \oplus \wedge^1$. We denote

$$D(A) = \{(f_1, f_0) \in \mathrm{Der}(A) \oplus \mathrm{Der}_L(A) \mid (3.3) \text{ holds for all } a, x \in A\}.$$

Thus $\mathrm{HQ}^1(A)$ is computed as follows by definition.

Proposition 3.12. *Keep the above notation. Then we have* $\mathrm{HQ}^1 = D(A)/\mathrm{Im}\,\sigma^0$ *and hence*

$$\dim_{\Bbbk} \mathrm{HQ}^1(A) = \dim_{\Bbbk} D(A) - \dim_{\Bbbk} A + \dim_{\Bbbk} \mathrm{HQ}^0(A).$$

4. Examples

Example 4.1 (Standard Poisson algebras). Let A be an associative algebra. For any $a, b \in A$, we denote by $[a, b]$ the commutator $ab - ba$ of a and b. Then $(A, \cdot, \lambda[-,-])$ is a Poisson algebra for a fixed scalar $\lambda \in \Bbbk$, and we call it a *standard Poisson algebra*. By Proposition 3.11, we have $\mathrm{HQ}^0(A) = Z(A)$.

More generally, $\mathrm{HQ}^0(A) = Z(A)$ for any inner Poisson algebra since $Z(A) \subset Z\{A\}$ in this case, see [19, Lemma 1.1] for more details. Recall that a Poisson algebra $(A, \cdot, \{-,-\})$ is said to be inner if the Hamilton operator $\mathrm{ham}(a) := \{a, -\}$ is an inner derivation of $(A, \cdot)$ for any $a \in A$.

Now we turn to $\mathrm{HQ}^1(A)$. Given $f_1 \in \mathrm{Der}(A)$ and $f_0 \in \mathrm{Der}_L(A)$. Note that in standard case, the equality (3.3) is equivalent to

$$\mathrm{Im}(f_0 - f_1) \subseteq Z(A),$$

which holds if and only if $f_1 = f_0 + g$ for some Lie derivation g satisfying $\mathrm{Im}\, g \subseteq Z(A)$. Since $g([x, y]) = [g(x), y] + [x, g(y)]$, we have $\mathrm{Ker}(g) \supseteq [A, A]$, thus g induces some $\widetilde{g} \in \mathrm{Hom}(A/[A, A], Z(A))$. Conversely, each $\widetilde{g} \in \mathrm{Hom}(A/[A, A], Z(A))$ gives a Lie derivation g with $\mathrm{Im}\, g \subseteq Z(A)$. Thus we have the following characterization.

Proposition 4.2. *Let A be a standard Poisson algebra. Then*

$$\mathrm{HQ}^1(A) \cong \mathrm{HH}^1(A) \oplus \mathrm{Hom}(A/[A, A], Z(A)).$$

In general, quasi-Poisson cohomology groups of higher degrees are difficult to compute, and only some special cases are known to us.

Example 4.3. Let A be the $\Bbbk$-algebra of upper triangular 2×2 matrices. It is known to be the path algebra of the quiver of $\mathbb{A}_2$ type. More explicitly, A has a $\Bbbk$-basis $\{e_1, e_2, \alpha\}$, and the multiplication is given by $e_i e_j = \delta_{ij} e_i$, $\alpha e_1 = e_2 \alpha = 0$ and $e_1 \alpha = \alpha e_2 = \alpha$, where δ_{ij} is the Kronecker sign. Clearly $1_A = e_1 + e_2$.

Consider the standard Poisson algebra. By direct computation, one shows that as a graded algebra, $\mathrm{HQ}^*(A) \cong \Bbbk\langle x, y\rangle/\langle x^2, y^2, xy + yx\rangle$, the exterior algebra in two variables. The grading is given by $\deg(x) = \deg(y) = 1$.

Example 4.4. Consider the standard Poisson algebra of $A = \mathbb{M}_2(\Bbbk)$, the $\Bbbk$-algebra of 2×2 matrices. Again direct calculation shows that

$$\mathrm{HQ}^0(A) = \mathrm{HQ}^1(A) = \mathrm{HQ}^3(A) = \mathrm{HQ}^4(A) = \Bbbk,$$

and $\mathrm{HQ}^i = 0$ for $i \neq 0, 1, 3, 4$. In fact, as a graded algebra

$$\mathrm{HQ}^*(A) \cong \Bbbk\langle x, y\rangle/\langle x^2, y^2, xy + yx\rangle,$$

where the grading is given by $\deg(x) = 1$ and $\deg(y) = 3$.

Example 4.5 (Poisson algebras with trivial Lie bracket). Let $(A, \cdot, \{-,-\})$ be a finite-dimensional Poisson algebra with trivial Lie structure, i.e. $\{a,b\} = 0$ for any $a, b \in A$. Clearly, $\mathcal{Q}(A) = A \otimes A^{\mathrm{op}} \otimes \mathcal{U}(A)$ and $\mathcal{U}(A) \cong \mathcal{S}(A)$, where $\mathcal{S}(A)$ is the polynomial algebra of the vector space A.

One shows easily that as a $\mathcal{Q}(A)$-module, A is the tensor product of the A^e-module A and the trivial Lie module $\Bbbk$ over A. Then by a classical result in homological algebra, $\mathrm{HQ}^*(A) \cong \mathrm{HH}^*(A) \otimes \mathrm{Ext}^*_{\mathcal{S}(A)}(\Bbbk, \Bbbk)$; see for instance [2, Chap. XI, Theorem 3.1]. By Koszul duality, $\mathrm{Ext}^*_{\mathcal{S}(A)}(\Bbbk, \Bbbk) \cong \wedge A$, the exterior algebra of the vector space A. Thus we have the following result.

Proposition 4.6. *Let $(A, \cdot, \{-,-\})$ be a finite-dimensional Poisson algebra with the trivial Lie bracket. Then $\mathrm{HQ}^*(A) \cong \mathrm{HH}^*(A) \otimes \wedge A$.*

Example 4.7 (Poisson algebras with finite Hochschild cohomology dimension). Let $(A, \cdot, \{-,-\})$ be a Poisson algebra. Suppose that the associative algebra A has finite Hochschild cohomology dimension, that is, the nth Hochschild cohomology group of $(A, \cdot)$ vanishes for sufficiently large n.

Proposition 4.8. *Let $(A, \cdot, \{-,-\})$ be a Poisson algebra and k a fixed positive integer. Suppose $\mathrm{HH}^n(A) = 0$ for all $n > k$. Set $\Omega^k_n = \mathrm{Hom}(\bigoplus_{i+j=n, i \le k} A^i \otimes \wedge^j, A)$. Then the nth quasi-Poisson cohomology group*

$$\mathrm{HQ}^n(A) = \frac{\mathrm{Ker}\,\sigma^n \cap \Omega^k_n}{\mathrm{Im}\,\sigma^{n-1} \cap \Omega^k_n}.$$

Proof. To compute the quasi-Poisson cohomology, we use the quasi-Poisson complex $\mathrm{QC}(A, A)$ again. Consider the $\Bbbk$-linear map $\pi \colon \mathrm{Ker}\,\sigma^n \cap \Omega^k_n \to \mathrm{HQ}^n(A)$, $f \mapsto f + \mathrm{Im}\,\sigma^{n-1}$. Suppose $f = (f_n, \dots, f_1, f_0) \in \mathrm{Ker}\,\sigma^n$ for some $n > k$. By definition f_n is an nth cocycle in the Hochschild complex, and hence there exists some $g_{n-1} \in \mathrm{Hom}(A^{n-1}, A)$ such that $f_n = \delta^{n-1}(g_{n-1})$ since the nth Hochschild cohomology group vanishes, where δ^n is the nth differential in the Hochschild complex. Clearly, $\overline{f} = \overline{f - \sigma^{n-1} g} \in \mathrm{HQ}^n(A)$ with $g = (g_{n-1}, 0, \dots 0) \in \mathrm{Hom}(\bigoplus_{i=0}^{n-1} A^i \otimes \wedge^{n-1-i}, A)$. Thus,

$$f - \sigma^n(g) = (0, \widetilde{f}_{n-1}, f_{n-2}, \dots, f_0) \in \mathrm{Ker}\,\sigma^n.$$

For brevity, we still denote $\widetilde{f}_{n-1}$ by f_{n-1}. Therefore

$$\begin{aligned} & a_1 f_{n-1}(a_2 \otimes \cdots \otimes a_n \otimes x) \\ &\quad + \sum_{k=1}^{n-1} (-1)^i f_{n-1}(a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_n \otimes x) \\ &\quad + (-1)^n f_{n-1}(a_1 \otimes \cdots \otimes a_{n-1} \otimes x) a_n = 0. \end{aligned}$$

If $n - 1 > k$, we choose a $\Bbbk$-basis $\{v_i \mid i \in S\}$ for $\wedge^1 = A$ and define a $\Bbbk$-linear map $f^i_{n-1} \in \mathrm{Hom}(A^{n-1}, A)$ such that

$$f^i_{n-1}(a_1 \otimes \cdots \otimes a_{n-1}) = f_{n-1}(a_1 \otimes \cdots \otimes a_{n-1} \otimes v_i)$$

for each $i \in S$. Clearly, for each $i \in S$, f^i_{n-1} is an $(n-1)$th cocycle in the Hochschild complex. Since $\mathrm{HH}^{n-1}(A) = 0$, there exists a $\Bbbk$-linear map $g^i_{n-2} \in \mathrm{Hom}(A^{n-2}, A)$ such that

$$f^i_{n-1} = \delta^{n-2}(g^i_{n-2}).$$

Now $\{g^i_{n-2} \mid i \in S\}$ determines a $\Bbbk$-linear map $g_{n-2} \in \mathrm{Hom}(A^{n-2} \otimes \wedge^1, A)$ as follows. For any $a_1, \ldots, a_{n-2} \in A$ and any $i \in S$,

$$g_{n-2}(a_1 \otimes \cdots \otimes a_{n-2} \otimes v_i) = g^i_{n-2}(a_1 \otimes \cdots \otimes a_{n-2}).$$

By the construction of f^i_{n-1}'s and g_{n-2}, we have

$$\begin{aligned} &f_{n-1}(a_1 \otimes \cdots \otimes a_{n-1} \otimes x) \\ &\quad= \sum_{i \in S} \lambda_i f^i_{n-1}(a_1 \otimes \cdots \otimes a_{n-1}) \\ &\quad= \sum_{i \in S} \lambda_i \delta^{n-2}(g^i_{n-2})(a_1 \otimes \cdots \otimes a_{n-1}) \\ &\quad= (\sigma^{n-1}(g))_{n-1}(a_1 \otimes \cdots \otimes a_{n-1} \otimes x) \end{aligned}$$

for any $a_1 \otimes \cdots \otimes a_{n-1} \otimes x \in A^{n-1} \otimes \wedge^1$, where $x = \sum_{i \in S} \lambda_i v_i$, and $g = (0, g_{n-2}, 0, \ldots, 0) \in \mathrm{Hom}(\bigoplus_{i=0}^{n-1} A^i \otimes \wedge^{n-1-i}, A)$. So $\overline{f} = \overline{f - \sigma^{n-1}(g)} \in \mathrm{HQ}^{n-1}(A)$, and

$$f - \sigma^{n-1}(g) = (0, 0, \widetilde{f}_{n-2}, f_{n-3}, \ldots, f_0).$$

Denote again $\widetilde{f}_{n-2} = f_{n-2}$.

By repeating the above argument, we know that each $f \in \mathrm{HQ}^n(A)$ can be written as

$$\overline{f} = \overline{(0, \ldots, 0, f_k, \ldots, f_0)}.$$

Therefore, the $\Bbbk$-homomorphism π is surjective. Clearly, $\mathrm{Ker}\,\pi = \Omega^k_n \cap \mathrm{Im}\,\sigma^{n-1}$, and hence

$$\mathrm{HQ}^n(A) = \frac{\mathrm{Ker}\,\sigma^n \cap \Omega^k_n}{\mathrm{Im}\,\sigma^{n-1} \cap \Omega^k_n}. \qquad \square$$

5. A Grothendieck Spectral Sequence for Quasi-Poisson Cohomology

In this section, we construct a Grothendieck spectral sequence for smash product algebras, and apply it to the calculation of extensions of quasi-Poisson modules. As a special case, this Grothendieck spectral sequence exhibit a close relation among the quasi-Poisson cohomology, the Hochschild cohomology and the Lie algebra cohomology.

We begin with a general situation. Let H be a Hopf algebra over $\Bbbk$ with the co-multiplication Δ and the bijective antipode S. Let A be a module algebra over H and $A\#H$ be the smash product. If M, N are modules over $A\#H$, then

$\mathrm{Hom}_A(M, N)$ is an H-module with the action given by $(hf)(x) = \sum h_2 f(S^{-1}h_1 x)$ for $x \in M$. It is easy to show the natural isomorphism $\mathrm{Hom}_H(\Bbbk, \mathrm{Hom}_A(M, N)) \cong \mathrm{Hom}_{A\#H}(M, N)$. Thus, we have the following well-known lemma which is crucial in our calculation.

Lemma 5.1. *Keep the above notation. Then we have the natural isomorphism of bifunctors*

$$\mathrm{Hom}_H(\Bbbk, \mathrm{Hom}_A(-, -)) \cong \mathrm{Hom}_{A\#H}(-, -).$$

Proof. For any $A\#H$-modules M, N, the natural isomorphism

$$\mathrm{Hom}_H(\Bbbk, \mathrm{Hom}_A(M, N)) \xrightarrow{\simeq} \mathrm{Hom}_{A\#H}(M, N)$$

is given by $(\mathbb{1} \mapsto f) \mapsto f$. The verification is routine so we omit the details. $\square$

Applying the Grothendieck spectral sequence [14, Theorem 10.47], we obtain a spectral sequence for a smash product.

Lemma 5.2. *Keep the above notation. Then we have a spectral sequence*

$$\mathrm{Ext}^q_H(\Bbbk, \mathrm{Ext}^p_A(M, N)) \Rightarrow \mathrm{Ext}^{p+q}_{A\#H}(M, N).$$

Consequently, we obtain a Grothendieck spectral sequence which is handy in calculating quasi-Poisson cohomology groups.

Corollary 5.3. *Let $\mathcal{Q}(A)$ the quasi-Poisson enveloping algebra of the Poisson algebra A and M, N be modules over $\mathcal{Q}(A)$. Then we have a spectral sequence*

$$\mathrm{Ext}^q_{\mathcal{U}(A)}(\Bbbk, \mathrm{Ext}^p_{A^e}(M, N)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{Q}(A)}(M, N).$$

In particular, if we take $M = A$, then we obtain a spectral sequence connecting the Hochschild cohomology, the Lie algebra cohomology of A and the quasi-Poisson cohomology.

Theorem 5.4. *Let A be a Poisson algebra and N a quasi-Poisson A-module. Then we have a spectral sequence*

$$\mathrm{HL}^q(A, \mathrm{HH}^p(A, N)) \Rightarrow \mathrm{HQ}^{p+q}(A, N).$$

Corollary 5.5. *Let A be a Poisson algebra with $\mathrm{HH}^p(A) = 0$ for all $p > 1$. Then we have the short exact sequence*

$$0 \to \mathrm{HL}^{n-1}(A, \mathrm{Der}^{\circ}(A)) \to \mathrm{HQ}^n(A) \to \mathrm{HL}^n(A, Z(A)) \to 0$$

for $n \geq 1$, where $\mathrm{Der}^{\circ}(A)$ is the space of outer derivations of A.

Proof. By the assumption, we have $\mathrm{HH}^p(A) = 0$ for any $p \geq 2$, $\mathrm{HH}^1(A) = \mathrm{Der}^{\circ}(A)$ and $\mathrm{HH}^0(A) = Z(A)$. And by [6, Proposition 2.4], we have the following short exact sequence

$$0 \to \mathrm{HL}^{n-1}(A, \mathrm{Der}^{\circ}(A)) \to \mathrm{HQ}^n(A) \to \mathrm{HL}^n(A, Z(A)) \to 0$$

for $n \geq 1$. $\square$

Corollary 5.6. *Let A be a Poisson algebra over $\Bbbk$ with $\mathrm{HH}^p(A) = 0$ for all $p > 0$. Then $\mathrm{HQ}^*(A) \cong Z(A) \otimes \mathrm{HL}^*(A, \Bbbk)$ as graded algebras.*

Proof. By the assumption and Corollary 5.5, we have $\mathrm{Der}^{\mathrm{o}}(A) = 0$ and $\mathrm{HQ}^n(A) \cong \mathrm{HL}^n(A, Z(A))$. Furthermore, A has only inner Poisson structure and hence $Z(A) \subset Z\{A\}$. So $\mathrm{HQ}^*(A) \cong Z(A) \otimes \mathrm{HL}^*(A, \Bbbk)$. □

Corollary 5.7. *Let A be a finite-dimensional Poisson algebra. If A is homologically smooth as an associative algebra, then*

$$\mathrm{proj.dim}_{\mathcal{Q}(A)} A \leq \dim_{\Bbbk} A + \mathrm{proj.dim}_{A^e} A.$$

In particular, $\mathrm{HQ}^*(A)$ is finite-dimensional.

Proof. Recall that the homological smoothness of means $\mathrm{proj.dim}_{A^e} A < \infty$. Since A is finite-dimensional, we have $\wedge^q = 0$ and hence $\mathrm{HL}^q(A, -) = 0$ for any $q > \dim_{\Bbbk} A$. Therefore, $\mathrm{HL}^q(A, \mathrm{HH}^p(A, N)) = 0$ for $p > \mathrm{proj.dim}_{A^e} A$ or $q > \dim_{\Bbbk} A$. The desired inequality follows from the spectral sequence in Theorem 5.4.

In particular, $\mathrm{HQ}^n(A) = 0$ for sufficiently large n. Since A is finite-dimensional, $\mathrm{HQ}^n(A)$ is finite-dimensional for any $n \geq 0$, and the last conclusion follows. □

Example 5.8. Let Q be a finite connected quiver with underlying graph being a tree. Denote by $\Bbbk Q$ the path algebra of Q. Then we have $\mathrm{HH}^p(\Bbbk Q) = 0$ for any $p \geq 1$, see [7, Sec. 1.6]. We consider the standard Poisson structure on $\Bbbk Q$. By Corollary 5.6 it is immediate that $\mathrm{HQ}^n(\Bbbk Q) = \mathrm{HL}^n(\Bbbk Q, \Bbbk)$, the usual nth Lie algebra cohomology group of $(A, [-,-])$ with coefficients in $\Bbbk$.

Example 5.9. Let Q be the 2-Kronecker quiver and A be the path algebra of Q. Then we have $\mathrm{HH}^p(A) = 0$ for all $p \geq 2$, $\mathrm{HH}^1(A) = \Bbbk^3$, and $\mathrm{HH}^0(A) = \Bbbk$. By a direct calculation, we know that $\mathrm{HH}^1(A)$ is a trivial module over the Lie algebra $(A, [-,-])$. We consider the standard Poisson algebra of A. By some tedious calculations, we obtain $\mathrm{HL}^0(A, \Bbbk) = \Bbbk$, $\mathrm{HL}^1(A, \Bbbk) = \Bbbk^2$, $\mathrm{HL}^2(A, \Bbbk) = \Bbbk$, and $\mathrm{HL}^p(A, \Bbbk) = 0$ for any $p \geq 3$. From Corollary 5.5, it follows that $\mathrm{HQ}^0(A) = \Bbbk$, $\mathrm{HQ}^1(A) = \Bbbk^5$, $\mathrm{HQ}^2(A) = \Bbbk^7$, $\mathrm{HQ}^3(A) = \Bbbk^3$ and $\mathrm{HQ}^n(A) = 0$ for all $n \geq 4$.

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